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BRST quantization of string theory in AdS_3

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Abstract: We study the BRST quantization of bosonic and NSR strings propagating in $AdS_3 \times \mathcal{N}$ backgrounds. The no-ghost theorem is proved using the Frenkel-Garland-Zuckerman method. Regular and spectrally-flowed representations of affine $\mathrm{SL}(2,\mathbb{R})$ appear on an equal footing. Possible generalizations to related curved backgrounds are discussed.

Keywords: Conformal Field Models in String Theory, BRST Quantization, Superstring Vacua, Superstrings and Heterotic Strings.

Contents

1. Introduction

The BRST quantization of gauge theories was first applied to string theory in [1, 2], and in [3] a proof of the no-ghost theorem was given in the BRST context. Once the physical space was identified with the cohomology of the BRST operator at a particular ghost number, the cohomology itself is built more or less explicitly, in order to show that it does not contain states of negative norm. In particular, in [3] it was shown that, for strings in flat space, physical states can be isometrically identified with the excitations in the light-cone gauge.

In [4] Frenkel, Garland and Zuckerman (FGZ) presented a technically simple but conceptually deep new proof of the no-ghost theorem in the BRST context, which relies heavily on the fact that the *relative* BRST cohomology[1] at non-zero momentum is concentrated

[1]The relative BRST cohomology is defined imposing the additional constraints $b_0 = L_0 = 0$ to the BRST cohomolgy (for fermionic strings in the Ramond sector, also $\beta_0 = G_0 = 0$). See appendix A.

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at one ghost number,[2] a result known as the "vanishing theorem". Once the latter holds,[3] the proof of the no-ghost theorem is reduced to showing the equality between the traces of two operators (the method was discussed in [7]). The power of the FGZ method lies on its economy: it probes the unitarity of the physical sector without explicitly building it. The method was applied to bosonic strings in [4, 7], to NRS strings in [5], to $\mathbb{R}^{1,d-1} \times \mathcal{N}$ ($d \geq 2$) backgrounds in [8] and to $\mathbb{R}^1 \times \mathcal{N}$ backgrounds in [9].

The FGZ proof of the vanishing theorem [4] and its NSR extensions in [5], use techniques of homological algebra and do not depend on the details of the chiral algebra of the matter conformal field theory (CFT). The only assumption needed is that the matter CFT should be an L_--free Virasoro module.[4] For flat space worldsheet CFT, the latter was shown in [10].

In this work we will apply the FGZ method to bosonic and NSR strings propagating in $AdS_3 \times \mathcal{N}$ backgrounds [11, 12]. Here $\mathcal{N}$ is a compact unitary CFT, with the necessary central charge for the worldsheet CFT to be critical. These vacua are relevant in the context of the AdS/CFT correspondence [13]. In the AdS_3/CFT_2 case, the infinite modes of the two-dimensional (super)conformal algebras in the dual boundary CFT can be constructed explicitly (see [14, 15, 16, 17] and references therein).

The vanishing theorem holds in $AdS_3 \times \mathcal{N}$ vacua, because the worldsheet CFT is an L_--free Virasoro module, as shown in [18, 19] for regular $\mathrm{SL}(2,\mathbb{R})$ representations and in [11, 20] for the spectrally flowed cases.[5]

As in the $\mathbb{R}^1 \times \mathcal{N}$ vacua studied in [9], $AdS_3 \times \mathcal{N}$ is a nontrivial setting to see the BRST quantization at work since, due to the absence of a lightcone direction in AdS_3, we cannot identify the action of the ghost degrees of freedom as annihilating the "transversal" string excitations.

The unitarity puzzles posed by string theory in AdS_3 backgrounds [21] were settled in the old covariant quantization (OCQ) formalism in [18, 22, 19, 20]. It was shown there that physical states belong to the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset, modulo spurious states. This reduces the problem to the unitarity of the coset representations, which was proved in [23] and [20]

[2]As shown in [4] (and in [5] for the NSR string), the cohomology of the *full* BRST complex at nonzero momentum consists of two isomorphic copies at consecutive ghost numbers.

[3]It should be noted that the vanishing theorem does not hold in less conventional settings. For example, the BRST analysis of 2D gravity coupled to $c \leq 1$ matter shows that physical states are distributed through several ghost numbers [6].

[4]This means that the Hilbert space of the chiral CFT can be expressed as a direct sum of spaces, each of which is built by the action of Virasoro $L_{n<0}$ operators on a Virasoro highest-weight state, such that states with different $L_{n<0}$ content are linearly independent. In the superconformal case, the operators $G_{n<0}$ should be included. The importance of this condition can be seen in a hand-waving way as follows. Since the BRST charge Q is built out of the Virasoro modes only (and knows nothing about the worldsheet chiral algebra), certain homological statements about Q (such as the vanishing theorem), can be proved when a basis for the space upon which Q acts, can itself be obtained from Virasoro modes only.

[5]For fermionic strings, refs. [19, 20] consider only the NS sector. The results can be easily extended to the R sector. Alternatively, one can consider the $AdS_3 \times \mathcal{N}$ backgrounds as a particular case of the backgrounds studied in [9], the timelike $\mathrm{U}(1)$ current being the $J^3(z)$ current of $\mathrm{SL}(2,\mathbb{R})$. But since the whole $AdS_3 \times \mathcal{N}$ CFT is an L_--free Virasoro module, the new filtration introduced in [9] (which considers only the timelike $\mathrm{U}(1)$, $c = 1$ CFT), is not strictly necessary.

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for the bosonic and fermionic cases, respectively. Coset unitarity requires a bound for the the spin of the discrete representations of $\mathrm{SL}(2,\mathbb{R})$. This bound is rather ubiquitous, and we discuss it in section 3.1. As we show in section 4, one important point in which the OCQ formalism differs from the FGZ approach, is that for fermionic NRS strings, in the FGZ proof there is no need for the unitarity of the *supersymmetric* $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset [20], the bosonic results of [23] being sufficient. A different approach to string unitarity in AdS_3 has been advocated in [24].

Both in the OCQ and FGZ approaches, the general results do not hold when $J_0^3 = 0$, since the matter CFT is no longer an L_--free module. The unitarity of physical states must then be checked by hand.[6] These exceptional cases have been already studied in [11, 19, 20] for different representations, and will not be considered here.

The plan of this work is as follows. In section 2 we review the string spectrum in AdS_3. In section 3 we review the Hodge theory of the string BRST complex, which is an elegant and clear setting in which to formulate the FGZ method, which we do in section 3.1. In section 4 we calculate all the traces needed to prove the no-ghost theorem for the different representations. Finally, in section 5 we discuss possible generalizations to other curved backgrounds. In appendix A we summarize the essentials of the BRST formalism needed in this work. Appendices B and C are devoted to technical points used in the body of the paper. We work in the holomorphic sector. For closed strings, a similar antiholomorphic copy should be considered.

2. The affine $\mathrm{SL}(2,\mathbb{R})$ theory

2.1 The bosonic algebra

The space AdS_3 is the universal cover of the group manifold $\mathrm{SL}(2,\mathbb{R})$, so the action of a string moving in an AdS_3 background is an $\mathrm{SL}(2,\mathbb{R})$ WZW model. The symmetry algebra is the affine $\mathrm{SL}(2,\mathbb{R})$ algebra at level k ($k > 2$), generated by three currents $J^{3,\pm}(z)$ satisfying the OPEs

$$\begin{aligned} J^3(z)J^3(w) &\sim -\frac{k/2}{(z-w)^2}\,, \\ J^+(z)J^-(w) &\sim \frac{k}{(z-w)^2} + \frac{2J^3(w)}{z-w}\,, \\ J^3(z)J^\pm(w) &\sim \pm\frac{J^\pm(w)}{z-w}\,. \end{aligned} \tag{2.1}$$

Using the Sugawara construction, the stress tensor is

$$T^{\mathrm{SL}(2,\mathbb{R})} = \frac{1}{k-2}\frac{1}{2}\left(J^+J^- + J^+J^- - 2J^3J^3\right). \tag{2.2}$$

and its modes close a Virasoro algebra with central charge $c_{\mathrm{SL}(2,\mathbb{R})} = \frac{3k}{k-2}$. Expanding the currents in modes as

$$J^{3,\pm}(z) = \sum_{n\in\mathbb{Z}} \frac{J_n^{3,\pm}}{z^{n+1}}\,, \tag{2.3}$$

[6] This is similar to the zero-momentum states in flat space.

the current algebra (2.1) can be written as

$$\begin{aligned}
[J_n^3, J_m^3] &= -\frac{k}{2} n \delta_{n+m,0} \, , \\
[J_n^3, J_m^\pm] &= \pm J_{n+m}^\pm \, , \\
[J_n^+, J_m^-] &= -2J_{n+m}^3 + kn\delta_{n+m,0} \, ,
\end{aligned} \tag{2.4}$$

with all other commutators vanishing.

The highest weight representations of (2.1) and (2.4) are built starting from unitary representations of the $\mathrm{SL}(2,\mathbb{R})$ Lie algebra $J_0^{3,\pm}$. These are characterized by the eigenvalues of the Casimir operator $\frac{1}{2}(J_0^+ J_0^- + J_0^- J_0^+) - (J_0^3)^2$, which will be denoted by $-j(j-1)$, and the states within each representation are labelled by the eigenvalue of J_0^3, which will be denoted by m. These representations are the primary states for the current algebra. They are annihilated by $J_{n>0}^{3,\pm}$, and the Fock space of states is built by acting on them with $J_{n<0}^{3,\pm}$.

The unitary representations of the $\mathrm{SL}(2,\mathbb{R})$ Lie algebra $J_0^{3,\pm}$ which appear in the spectrum of strings moving in an $AdS_3 \times \mathcal{N}$ background are [11, 12]:

1. *Highest weight discrete representations*

   $$\mathcal{D}_j^+ = \{|j;m\rangle : m = j, j+1, j+2, \ldots\}$$

   where $|j;j\rangle$ is annihilated by J_0^- and j is a real number such that $1/2 < j < (k-1)/2$.

2. *Lowest weight discrete representations*

   $$\mathcal{D}_j^- = \{|j;m\rangle : m = -j, -j-1, -j-2, \ldots\}$$

   where $|j;-j\rangle$ is annihilated by J_0^+ and j is a real number such that $1/2 < j < (k-1)/2$.

3. *Continuous representations*

   $$\mathcal{C}_{j,\alpha} = \{|j,\alpha;m\rangle : m = \alpha, \alpha \pm 1, \alpha \pm 2, \ldots\}$$

   where $0 \leq \alpha < 1$ and $j = 1/2 + is$, where s is a real number.

The bounds on j appearing for $\mathcal{D}_j^\pm$ can be understood in terms of consistency conditions for the primary states. The lower bound, $1/2 < j$, is necessary for the normalizability of the primary states when their norm is interpreted as the $\mathcal{L}^2$ inner product of functions in the $\mathrm{SL}(2,\mathbb{R})$ group manifold [25]. Regarding the upper bound, $j < (k-1)/2$, in [26] it was noted that it is necessary for the unitarity of the primary states, when their norm is interpreted as the two-point function of the vertex operators creating them from the vacuum. Moreover, adopting either the upper or the lower bound for j, the other one appears when imposing the $w = \pm 1$ spectral flow (see below) to be a symmetry of the spectrum. Finally, the compelling evidence for the correctness of these bounds on j comes from the fact that only this range of j appears in the spectrum of the thermal partition function of the model, computed by path integral techniques in [12].

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In the computations of section 4, we will be interested in the characters of the representations [11], which are very easy to obtain because the action of the $J^{3,\pm}_{n<0}$ modes is free for the representations considered [27]. For $\mathcal{D}^+_j$ we have

$$\begin{aligned}
\chi^+_j(q,z) = \mathrm{Tr}_{\mathcal{D}^+_j}\left(q^{L_0} z^{J^3_0}\right) &= \frac{q^{-\frac{j(j-1)}{k-2}} z^j}{\prod_{n=1}^\infty (1-q^n)(1-q^{n-1}z)(1-q^n z^{-1})}\,, \\
&= \frac{q^{-\frac{j(j-1)}{k-2}+\frac{1}{8}} z^{j-\frac{1}{2}}}{i\Theta_1(q,z)}\,.
\end{aligned} \tag{2.5}$$

Eq. (2.5) should be understood as a formal series expansion which converges to (2.5) when $|q| < |z| < 1$. The elliptic function $\Theta_1(q,z)$ is [28]

$$\begin{aligned}
\Theta_1(q,z) &= -iz^{-1/2} q^{1/8} \prod_{n=1}^\infty (1-q^n)(1-q^{n-1}z)(1-q^n z^{-1})\,, \\
&= -i \sum_{n\in\mathbb{Z}} (-1)^n z^{n-1/2} q^{\frac{(n-1/2)^2}{2}}\,.
\end{aligned} \tag{2.6}$$

For the $\mathcal{C}_{j,\alpha}$ representations, the character is

$$\hat{\chi}_{j=1/2+is,\alpha}(q,z) = \mathrm{Tr}_{\mathcal{C}_{j=1/2+is,\alpha}}\left(q^{L_0} z^{J^3_0}\right) = \frac{q^{\frac{s^2+1/4}{k-2}}}{\prod_{n=1}^\infty (1-q^n)^3} \sum_{n\in\mathbb{Z}} z^{\alpha+n}\,. \tag{2.7}$$

The character of $\mathcal{D}^-_j$ can be obtained from the spectral flow of $\mathcal{D}^+_j$ (see below).

2.2 SL(2, ℝ)/ U(1) × U(1) decomposition and spectral flow

It will be convenient to decompose both the currents and the spectrum into the direct product of a parafermionic SL(2, ℝ)/ U(1) model and the timelike U(1) model corresponding to $J^3(z)$. For the currents $J^{3,\pm}(z)$, this can be performed by representing them as

$$\begin{aligned}
J^3(z) &= -\sqrt{\frac{k}{2}}\partial X(z)\,, \\
J^\pm(z) &= \psi^\pm(z) e^{\pm\sqrt{\frac{2}{k}}X}\,,
\end{aligned} \tag{2.8}$$

with

$$\begin{aligned}
X(z)X(w) &\sim -\log(z-w)\,, \\
X(z)\psi^\pm(w) &\sim 0\,.
\end{aligned} \tag{2.9}$$

Note that the field $X(z)$ is antihermitean, in order for $J^3(z)$ to be hermitean and $J^+(z)^\dagger = J^-(z)$. The fields $\psi^\pm$ form an SL(2, ℝ)/ U(1) parafermionic theory introduced in [29], and are a non-minimal generalization of the SU(2)/ U(1) parafermions of [30]. The OPEs of $\psi^\pm$ can be read from (2.1) and (2.9).

The stress tensor is similarly decomposed into

$$T^{\mathrm{SL}(2,\mathbb{R})} = T^{\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)} + T^{\mathrm{U}(1)}\,, \tag{2.10}$$

with

$$T^{\mathrm{U}(1)} = -\frac{1}{k} J^3 J^3 \,. \tag{2.11}$$

The decomposition can also be performed on the representations. Indeed, every representation $\mathcal{D}_j^+, \mathcal{C}_j$ gives rise to an infinite number of parafermionic representations $\lambda_{j,n}, \hat{\lambda}_{j,n}$ with $n \in \mathbb{Z}$. The $\mathcal{D}_j^+, \mathcal{C}_j$ Hilbert spaces can be decomposed into

$$\begin{aligned} \mathcal{D}_j^+ &= \sum_{n\in\mathbb{Z}} \lambda_{j,n} \otimes \mathcal{U}_{j+n} \,, \\ \mathcal{C}_{j,\alpha} &= \sum_{n\in\mathbb{Z}} \hat{\lambda}_{j,\alpha+n} \otimes \mathcal{U}_{\alpha+n} \,, \end{aligned} \tag{2.12}$$

where $\mathcal{U}_m$ is a highest-weight representation of the timelike current $J^3(z)$ with J_0^3 eigenvalue m. The $\lambda_{j,n}, \hat{\lambda}_{j,n}$ representations are built by acting with the modes of $\psi^\pm$ on parafermionic highest-weight states. The reader can find details in [29, 30, 31, 32]. We will be mainly interested in their characters [31, 32], which are

$$\begin{aligned} \lambda_{j,n}(q) &= \mathrm{Tr}_{\lambda_{j,n}}\big(q^{L_0}\big) = \frac{q^{-\frac{j(j-1)}{k-2}+\frac{(j+n)^2}{k}}}{\prod_{n=1}^\infty (1-q^n)^2} \sum_{s=0}^\infty (-1)^s q^{\frac{1}{2}s(s+2n+1)} \,, \\ \hat{\lambda}_{j,\alpha+n}(q) &= \mathrm{Tr}_{\hat{\lambda}_{j,\alpha+n}}\big(q^{L_0}\big) = \frac{q^{-\frac{j(j-1)}{k-2}+\frac{(\alpha+n)^2}{k}}}{\prod_{n=1}^\infty (1-q^n)^2} \,. \end{aligned} \tag{2.13}$$

In appendix A we show explicitly that the decompositions (2.12) are, in terms of the characters,

$$\begin{aligned} \chi_j^+(q,z) &= \sum_{n\in\mathbb{Z}} z^{j+n}\, \lambda_{j,n}(q)\, \zeta_{j+n}(q) \,, \\ \hat{\chi}_{j=1/2+is,\alpha}(q,z) &= \sum_{n\in\mathbb{Z}} z^{\alpha+n}\, \hat{\lambda}_{j,\alpha+n}(q)\, \zeta_{\alpha+n}(q) \,, \end{aligned} \tag{2.14}$$

where

$$\zeta_m(q) \equiv \mathrm{Tr}_{\mathcal{U}_m}\left(q^{L_0}\right) = \frac{q^{-\frac{m^2}{k}}}{\prod_{n=1}^\infty (1-q^n)} \,. \tag{2.15}$$

Now, given any integer w, the algebra (2.4) is preserved by the spectral flow $J_n^{3,\pm} \to \tilde{J}_n^{3,\pm}$ first introduced in [33] and defined by

$$\begin{aligned} \tilde{J}_n^3 &= J_n^3 - \frac{k}{2} w \delta_{n,0} \,, \\ \tilde{J}_n^\pm &= J_{n\pm w}^\pm \,. \end{aligned} \tag{2.16}$$

This transformation maps L_0 to

$$\tilde{L}_0 = L_0 + w J_0^3 - \frac{k}{4} w^2 \,. \tag{2.17}$$

The spectral flow can be implemented in (2.8) by

$$X(z) \longrightarrow \tilde{X}(z) = X(z) + w\sqrt{\frac{k}{2}} \log z \,, \tag{2.18}$$

and leaving the parafermions $\psi^{\pm}(z)$ untouched. For $w = \pm 1$, the symmetry (2.16) maps the $\mathcal{D}_j^{\mp}$ representation into $\mathcal{D}_{k/2-j}^{\pm}$. Note that under this mapping the upper and lower bounds of j are interchanged. Including the $w = \pm 1$ spectrally flowed states, we can just consider $\mathcal{D}_j^{+}$ representations.

For generic w, it was shown in [11, 12] that the Hilbert space $\mathcal{D}_j^{+}$ is mapped into new representations which must be included in a consistent description of a string propagating in $AdS_3 \times \mathcal{N}$. The way to include these representations is to take usual representations $\mathcal{D}_j^{+}, \mathcal{C}_{j,\alpha}$ for flowed operators $\tilde{J}_n^{3,\pm}$, and measure their quantum numbers with unflowed J_0^3, L_0. Alternatively, instead of this *active* procedure, we can use a *passive* one, and obtain the flowed sectors by acting with the unflowed operators $J_n^{3,\pm}$ on flowed primaries. Starting with an $\mathrm{SL}(2,\mathbb{R})$ vertex operator

$$\Phi_{j,m} = \Psi_{j,m} e^{m\sqrt{\frac{2}{k}}X} , \tag{2.19}$$

where $\Psi_{j,m}$ is a parafermionic primary, the flowed vertex operator is obtained applying to it the mapping (2.18), yielding

$$\Phi_{j,m}^{w} = \Psi_{j,m} e^{(m+wk/2)\sqrt{\frac{2}{k}}X} . \tag{2.20}$$

Note that for $\mathcal{C}_{j,\alpha}$ representations, all the excited physical states belong to flowed sectors. This can be seen from the mass-shell condition,

$$L_0 = \frac{\frac{1}{4} + s^2}{k-2} - w(\alpha + n) + \frac{kw^2}{4} + N + h^{\mathcal{N}} - 1 = 0 , \tag{2.21}$$

where $h^{\mathcal{N}}$ is a highest weight of the inner theory and N is the level. It is clear that for $w = 0$, eq. (2.21) can only be satisfied if $N = 0$.

The characters of the flowed $\mathcal{D}_j^{w}$ representations are

$$\begin{aligned} \chi_j^{w}(q,z) &= \mathrm{Tr}_{\mathcal{D}_j^{w}}\left(q^{L_0} z^{J_0^3}\right) = \mathrm{Tr}_{\mathcal{D}_j^{+}}\left(q^{\tilde{L}_0 - w\tilde{J}_0^3 - kw^2/4} z^{\tilde{J}_0^3 + kw/2}\right) \\ &= q^{-kw^2/4} z^{kw/2}\, \mathrm{Tr}_{\mathcal{D}_j^{+}}\left(q^{\tilde{L}_0} (zq^{-w})^{\tilde{J}_0^3}\right) \\ &= \frac{(-1)^w q^{\left[-\frac{j(j-1)}{k-2} - (k-2)\frac{w^2}{4} - (2j-1)\frac{w}{2} + \frac{1}{8}\right]} z^{\left[j + (k-2)\frac{w}{2} - \frac{1}{2}\right]}}{i\Theta_1(q,z)} . \end{aligned} \tag{2.22}$$

where we have used the property [28]

$$\Theta_1(q, zq^{-w}) = (-1)^w q^{-w^2/2} z^w \Theta_1(q,z) . \tag{2.23}$$

For the $\mathcal{C}_{j,\alpha}^{w}$ representations we obtain similarly

$$\begin{aligned} \hat{\chi}_{j=1/2+is,\alpha}^{w}(q,z) &= \mathrm{Tr}_{\mathcal{C}_{j,\alpha}^{w}}\left(q^{L_0} z^{J_0^3}\right) \\ &= \frac{q^{\frac{s^2+1/4}{k-2} - kw^2/4 - w\alpha}}{\prod_{n=1}^{\infty}(1-q^n)^3} \sum_{n\in\mathbb{Z}} z^{\alpha+n+kw/2} q^{-wn} . \end{aligned} \tag{2.24}$$

As is clear from (2.20), the effect of the spectral flow is to "dislocate" the parafermionic and the J_0^3 quantum numbers by an amount $wk/2$. The decompositions (2.12) are generalized to

$$\begin{aligned}
\mathcal{D}_j^w &= \sum_{n\in\mathbb{Z}} \lambda_{j,n} \otimes \mathcal{U}_{j+n+kw/2}\,, \\
\mathcal{C}_{j,\alpha}^w &= \sum_{n\in\mathbb{Z}} \hat{\lambda}_{j,\alpha+n} \otimes \mathcal{U}_{\alpha+n+kw/2}\,.
\end{aligned} \tag{2.25}$$

In appendix C we show explicitly that, in terms of the characters, the decompositions (2.25) are

$$\begin{aligned}
\chi_j^w(q,z) &= \sum_{n\in\mathbb{Z}} z^{j+n+kw/2}\, \lambda_{j,n}(q)\, \zeta_{j+n+kw/2}(q)\,, \\
\hat{\chi}_{j=1/2+is,\alpha}^w(q,z) &= \sum_{n\in\mathbb{Z}} z^{\alpha+n+kw/2} \hat{\lambda}_{j,\alpha+n}(q)\, \zeta_{\alpha+n+kw/2}(q)\,.
\end{aligned} \tag{2.26}$$

2.3 The supersymmetric algebra

For fermionic strings in $AdS_3 \times \mathcal{N}$, we should consider the current algebra of affine supersymmetric SL(2,$\mathbb{R}$) at level k. It is generated by three dimension-$\frac{1}{2}$ supercurrents $\psi^{3,\pm} + \theta J^{3,\pm}$, whose modes satisfy

$$\begin{aligned}
[J_n^3, J_m^3] &= -\frac{k}{2} n \delta_{n+m,0}\,, \\
[J_n^3, J_m^\pm] &= \pm J_{n+m}^\pm\,, \\
[J_n^+, J_m^-] &= -2J_{n+m}^3 + kn\delta_{n+m,0}\,, \\
[J_n^3, \psi_m^\pm] &= \pm \psi_{n+m}^\pm\,, \\
[J_n^\pm, \psi_m^\mp] &= \mp 2\psi_{n+m}^3\,, \\
[J_n^\pm, \psi_m^3] &= \mp \psi_{n+m}^\pm\,, \\
\{\psi_n^3, \psi_m^3\} &= -\frac{k}{2}\delta_{n+m,0}\,, \\
\{\psi_n^+, \psi_m^-\} &= k\delta_{n+m,0}\,,
\end{aligned} \tag{2.27}$$

with all other (anti)commutators vanishing. The modding of J_n^a is integer and that of ψ_n^a half-integer (Neveu-Schwarz) or integer (Ramond). The worldsheet CFT is supersymmetric, the Sugawara stress tensor and the supercurrent being

$$\begin{aligned}
T^{\mathrm{SL}(2,\mathbb{R})} &= \frac{1}{2k}[j^+j^- + j^-j^+] - \frac{1}{k}j^3j^3 - \frac{1}{2k}[\psi^+\partial\psi^- + \psi^-\partial\psi^+] + \frac{1}{k}\psi^3\partial\psi^3\,, \\
G^{\mathrm{SL}(2,\mathbb{R})} &= \frac{1}{k}[\psi^+j^- + \psi^-j^+] - \frac{2}{k}\psi^3j^3 - \frac{2}{k^2}\psi^+\psi^-\psi^3\,.
\end{aligned} \tag{2.28}$$

As usual, purely bosonic currents can be defined by

$$\begin{aligned}
j^a &= J^a - \hat{j}^a\,, \\
\hat{j}^a &= -\frac{i}{k} f^a{}_{bc}\psi^b\psi^c\,.
\end{aligned} \tag{2.29}$$

The j^a form an affine $SL(2,\mathbb{R})$ bosonic algebra at level $k+2$. The $\hat{J}^a$ and the ψ^a form a supersymmetric affine $SL(2,\mathbb{R})$ algebra at level -2 which commutes with j^a. The spectrum of the theory is the direct product of the Hilbert spaces of both theories. In the bosonic sector, the reprentations are the ones considered above. For the ψ^a currents, we have the usual representations for free fermions, for both Neveu-Schwarz and Ramond sectors. The spectral flow can be also defined in the $\psi^a, \hat{J}^a$ theory, but it just amounts to a rearrangement of the usual spectrum, as shown in [20].

3. The Hodge theory of the BRST complex

The formalism reviewed below was developed in [34], and can be found in [35]. For similar results in the context of compact Lie algebras, see [36].

We will denote by $\mathcal{F}$ the Hilbert space composed of the ghost plus matter sectors, with the restriction $b_0 = L_0 = 0$, and also $\beta_0 = G_0 = 0$ for fermionic strings in the Ramond sector. Although $\mathcal{F}$ is infinite dimensional (due to the infinite degeneracy of the base states of the $SL(2,\mathbb{R})$ representations), it can be decomposed into

$$\mathcal{F} = \bigoplus_m \mathcal{F}(m)\,, \tag{3.1}$$

where each subspace $\mathcal{F}(m)$ has $J_0^3 = m$ and is finite dimensional. In the following, we will consider each $\mathcal{F}(m)$ space separately, although we will keep the notation $\mathcal{F}$.

The space $\mathcal{F}$ has a natural grading under N_g as

$$\mathcal{F} = \bigoplus_{g\in\mathbb{Z}} \mathcal{F}_g\,, \tag{3.2}$$

where $\mathcal{F}_g$ has ghost number g. The action induced by Q in each $\mathcal{F}_g$ will de denoted by

$$Q_g : \mathcal{F}_g \to \mathcal{F}_{g+1}\,, \tag{3.3}$$

and the following sequence is a graded complex

$$\cdots \longrightarrow \mathcal{F}_{g-1} \xrightarrow{Q_{g-1}} \mathcal{F}_g \xrightarrow{Q_g} \mathcal{F}_{g+1} \longrightarrow \cdots\,. \tag{3.4}$$

For each g we define the kernel and the image of Q as

$$\begin{aligned} \operatorname{Ker} Q_g &= \{\xi \in \mathcal{F}_g \,|\, Q\xi = 0\}\,, \\ \operatorname{Im} Q_g &= \{Q\xi \,|\, \xi \in \mathcal{F}_g\}\,. \end{aligned} \tag{3.5}$$

The cohomology of Q in $\mathcal{F}$ is then defined as

$$\mathrm{H} = \bigoplus_g \mathrm{H}_g\,, \tag{3.6}$$

with

$$\mathrm{H}_g = \frac{\operatorname{Ker} Q_g}{\operatorname{Im} Q_{g-1}}\,. \tag{3.7}$$

Since the indefinite inner product in $\mathcal{F}$ is non-degenerate, an orthogonal basis in $\mathcal{F}$ can be constructed. Let us define the action of a linear hermitean operator $\mathcal{C}$ on the states of such a basis as 1 (-1) if the state has positive (negative) norm. Then for any pair of states $\xi, \eta \in \mathcal{F}$ the inner product

$$\langle \xi, \eta \rangle_{\mathcal{C}} \equiv \langle \xi, \mathcal{C}\eta \rangle \,, \tag{3.8}$$

is clearly positive definite. The latter implies that $\mathcal{C}$ maps $\mathcal{F}_g$ into $\mathcal{F}_{-g}$ because the anti-hermitean character of N_g implies that two states can have nonzero $\langle , \rangle$ inner product only if they have opposite ghost number. Indeed, $\mathcal{C}$ interchanges the b_n and the c_n modes of any state (and the β_n and γ_n for fermionic strings), as we show explicitly in appendix B. Moreover, this mapping from $\mathcal{F}_g$ to $\mathcal{F}_{-g}$ is a linear isomorphism because each $\mathcal{F}_g$ is finite dimensional and $\mathcal{C}$ has empty kernel because $\mathcal{C}^2 = 1$. Note that the action of $\mathcal{C}$ is similar to the action of the Hodge $*$ operator on p-forms.

Under this new inner product Q is no longer hermitean, but its conjugate Q^* can be obtained from

$$\langle \xi, Q\eta \rangle_{\mathcal{C}} = \langle \xi, \mathcal{C}Q\eta \rangle = \langle \mathcal{C}Q\mathcal{C}\xi, \mathcal{C}\eta \rangle = \langle Q^*\xi, \eta \rangle_{\mathcal{C}} \,, \tag{3.9}$$

where we have used $\mathcal{C}^2 = 1$, so we have $Q^* = \mathcal{C}Q\mathcal{C}$. From its definition it is clear that $(Q^*)^2 = 0$ and that Q^* carries ghost number -1 and is hermitean under $\langle , \rangle$. As for Q, we will denote by Q^*_g the action of Q^* on $\mathcal{F}_g$.

The positive definiteness of (3.8) allows us to decompose $\mathcal{F}_g$ both as

$$\mathcal{F}_g = \operatorname{Im} Q_{g-1} \oplus (\operatorname{Im} Q_{g-1})^{\perp} \,, \tag{3.10}$$

and as

$$\mathcal{F}_g = \operatorname{Im} Q^*_{g+1} \oplus (\operatorname{Im} Q^*_{g+1})^{\perp} \,, \tag{3.11}$$

where $(\cdots)^{\perp}$ means orthogonal with respect to $\langle , \rangle_{\mathcal{C}}$. But, as can be easily checked, we have $(\operatorname{Im} Q^*_{g+1})^{\perp} = \operatorname{Ker} Q_g$ and $(\operatorname{Im} Q_{g-1})^{\perp} = \operatorname{Ker} Q^*_g$. Applying now the decomposition (3.10) to $\operatorname{Ker} Q_g$ we have

$$\operatorname{Ker} Q_g = \operatorname{Im} Q_{g-1} \oplus \mathcal{H}_g \,, \tag{3.12}$$

where we have defined

$$\mathcal{H}_g = \operatorname{Ker} Q_g \cap \operatorname{Ker} Q^*_g \,. \tag{3.13}$$

The elements of $\mathcal{H}_g$ provide us with a unique representative for each cohomology class H_g. Indeed, from (3.13) we see that the elements of $\mathcal{H}_g$ are annihilated by Q_g, and two representatives of the same class of H_g always differ by an element belonging entirely to the first term of (3.12). Moreover, it is easy to check (using that $\langle , \rangle_{\mathcal{C}}$ is positive definite) that a state belongs to $\mathcal{H}_g$ iff it is *harmonic*, i.e., it is annihilated by the *laplacian*

$$\Delta \equiv QQ^* + Q^*Q \,. \tag{3.14}$$

Inserting now (3.12) into (3.11) we obtain the *Hodge decomposition*

$$\mathcal{F}_g = \operatorname{Im} Q_{g-1} \oplus \operatorname{Im} Q^*_{g+1} \oplus \mathcal{H}_g \,. \tag{3.15}$$

It is easy to check that the isomorphism between $\mathcal{F}_g$ and $\mathcal{F}_{-g}$ provided by $\mathcal{C}$ acts on the terms of (3.15) as

$$\begin{aligned}\mathcal{C}:&\ \ \mathrm{Im}\, Q_g \longleftrightarrow \mathrm{Im}\, Q^*_{-g}\,,\\ \mathcal{C}:&\ \ \mathcal{H}_g \longleftrightarrow \mathcal{H}_{-g}\,,\end{aligned} \tag{3.16}$$

and from the second line of (3.16) we find that the cohomologies H_g and H_{-g} are isomorphic, a result known as *Poincaré duality*.

Now, it is clear that

$$\dim \mathcal{F}_g = \dim \mathrm{Ker}\, Q_g + \dim \mathrm{Im}\, Q_g\,, \tag{3.17}$$

and from (3.12) we have $\dim \mathrm{Ker}\, Q_g = \dim \mathrm{Im}\, Q_{g-1} + \dim \mathcal{H}_g$, which implies

$$\dim \mathcal{F}_g = \dim \mathcal{H}_g + \dim \mathrm{Im}\, Q_g + \dim \mathrm{Im}\, Q_{g-1}\,. \tag{3.18}$$

Performing an alternating sum of (3.18) over all g, the last two terms cancel pairwise and we obtain the *Euler-Poincaré identity*:

$$\sum_g (-1)^g \dim \mathcal{F}_g = \sum_g (-1)^g \dim \mathcal{H}_g\ . \tag{3.19}$$

3.1 The FGZ proof of the no-ghost theorem

The physical space of states is defined to be the Q cohomology in $\mathcal{F}$. Since the latter is concentrated at zero ghost number, it can be identified with $\mathcal{H}_0$, and the no-ghost theorem asserts that states in $\mathcal{H}_0$ have positive norm under the usual inner product $\langle , \rangle$.

Note that from the second line of (3.16) we see that $\mathcal{C}$ maps $\mathcal{H}_0$ to itself, so it can be diagonalized in $\mathcal{H}_0$ with eigenvalues $+1$ or -1 (because $\mathcal{C}^2 = 1$), corresponding to states with positive or negative $\langle , \rangle$ norm, respectively. Calling N_+ (N_-) to the number of $+1$ (-1) eigenvalues, we have

$$\dim \mathcal{H}_0 \geq N_+ - N_- = \mathrm{Tr}_{\mathcal{H}_0}\, \mathcal{C}\,, \tag{3.20}$$

and it is clear that the no-ghost theorem is proved iff the equality in (3.20) holds.

The centrality of the vanishing theorem comes now to the fore, since having $\mathrm{H}_{g\neq 0} = 0$ implies, through the Euler-Poincaré identity (3.19),

$$\mathbb{E} \equiv \sum_g (-1)^g \dim \mathcal{F}_g = \sum_g (-1)^g \dim \mathcal{H}_g = \dim \mathcal{H}_0\,, \tag{3.21}$$

where $\mathbb{E}$ is the *Euler index* of the BRST complex. As for $\mathrm{Tr}_{\mathcal{H}_0}\, \mathcal{C}$, notice that the isomorphism (3.16) under $\mathcal{C}$ between $\mathrm{Im}\, Q_{-1}$ and $\mathrm{Im}\, Q_1^*$ implies

$$\mathrm{Tr}_{\{\mathrm{Im}\, Q_{-1} \oplus \mathrm{Im}\, Q_1^*\}}\, \mathcal{C} = 0\,, \tag{3.22}$$

so from (3.15) we have

$$\mathrm{Tr}_{\mathcal{H}_0}\, \mathcal{C} = \mathrm{Tr}_{\mathcal{F}_0}\, \mathcal{C}\,. \tag{3.23}$$

Similarly, the isomorphism under $\mathcal{C}$ between $\mathcal{F}_g$ and $\mathcal{F}_{-g}$ implies for $g \neq 0$ that

$$\mathrm{Tr}_{\{\mathcal{F}_g \oplus \mathcal{F}_{-g}\}}\, \mathcal{C} = 0\,, \tag{3.24}$$

and from this we obtain

$$\mathbb{S} \equiv \mathrm{Tr}_{\mathcal{F}}\, \mathcal{C} = \mathrm{Tr}_{\bigoplus_g \mathcal{F}_g}\, \mathcal{C} = \mathrm{Tr}_{\mathcal{H}_0}\, \mathcal{C}\,, \tag{3.25}$$

where $\mathbb{S}$ is the *signature* of the BRST complex. Putting (3.20)–(3.25) together, the no-ghost theorem is equivalent to the identity

$$\mathbb{S} = \mathbb{E}. \tag{3.26}$$

4. No-ghost theorem for strings in AdS_3

We will prove the identity (3.26) by comparing

$$\begin{aligned} \mathbb{S}(q,z) &= \mathrm{Tr}\left(\mathcal{C} q^{L_0} z^{J_0^3}\right), \\ \mathbb{E}(q,t) &= \mathrm{Tr}\left((-1)^{N_g} q^{L_0} z^{J_0^3}\right), \end{aligned} \tag{4.1}$$

where the traces will be taken over the different representations. Note that for our purposes the formal expansions in z and q are of very different nature. Because of the mass shell condition $L_0 = 0$, in the expansion in q we are only interested in the term independent of q.[7] On the other hand, we need all the powers of z, since the decomposition (3.1) was only done in order to separate $\mathcal{F}$ into finite-dimensional subspaces.

In the computation of (4.1), we will use the fact that the trace is multiplicative under tensor products and additive under direct sums. We will consider representations with arbitrary flow parameter w, which of course include the regular representations ($w = 0$).

4.1 Bosonic strings

In [23] it was shown that $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ representations coming from $\mathcal{C}_{j,\alpha}$ representations with $k > 2$, and those coming from $\mathcal{D}_j$ with $k > 2, j < k/2$, are unitary. Since these bounds are satisfied in our case, the only source of negative norms are the modes of $J^3(z)$. The action of $\mathcal{C}$ in the $\mathrm{SL}(2,\mathbb{R})$ sector should clearly be

$$\begin{aligned} \mathcal{C}\psi^{\pm}(z)\mathcal{C} &= \psi^{\pm}(z)\,, \\ \mathcal{C}J^3_{-n}\mathcal{C} &= -J^3_{-n}\,. \end{aligned} \tag{4.2}$$

We decompose $\mathcal{F}$ into

$$\mathcal{F} = \mathcal{F}_{(b,c)} \otimes \mathcal{F}_{\mathrm{SL}(2,\mathbb{R})} \otimes \mathcal{F}_{\mathcal{N}}\,, \tag{4.3}$$

and we further decompose the representations of $\mathcal{F}_{\mathrm{SL}(2,\mathbb{R})}$ as in (2.25)–(2.26).

[7] Note that the condition $b_0 = 0$ (and $\beta_0 = 0$) is already taken into account by just ignoring the degeneracy of the ghost vacuum in the traces (4.1).

The signature of the $\mathcal{D}_j^w$ representations can now be computed as

$$\mathbb{S}_{\mathcal{D}_j^w}(q,z) = \mathbb{S}_{(b,c)}(q) \times \sum_{n\in\mathbb{Z}} z^{j+n+kw/2} \mathbb{S}_{\lambda_{j,n}}(q) \mathbb{S}_{\mathcal{U}_{j+n+kw/2}}(q) \times \mathbb{S}_{\mathcal{N}}(q)\,. \tag{4.4}$$

The traces in (4.4) can be computed as follows. In the ghost sector, using the decomposition (B.1), we have

$$\mathbb{S}_{(b,c)}(q) = \mathrm{Tr}_{(b,c)}\left(\mathcal{C} q^{L_0}\right) = q^{-1} \prod_{n=1}^{\infty} (1-q^n)(1+q^n)\,. \tag{4.5}$$

The signature of the timelike U(1) is easily found from (4.2) to be

$$\mathbb{S}_{\mathcal{U}_{j+n+kw/2}}(q) = \mathrm{Tr}_{\mathcal{U}_{j+n+kw/2}}\left(\mathcal{C} q^{L_0}\right) = \frac{q^{-\frac{(j+n+kw/2)^2}{k}}}{\prod_{n=1}^{\infty}(1+q^n)}\,. \tag{4.6}$$

Since the theories $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ and $\mathcal{N}$ are unitary, their signature is their character. So from (4.4)–(4.6) the total signature is then

$$\mathbb{S}_{\mathcal{D}_j^w}(q,z) = q^{-1} \prod_{n=1}^{\infty}(1-q^n) \times \sum_{n\in\mathbb{Z}} q^{\frac{(j+n+kw/2)^2}{k}} z^{j+n+kw/2} \lambda_{j,n}(q) \times \mathrm{Tr}_{\mathcal{N}}\left(q^{L_0}\right). \tag{4.7}$$

The index of the $\mathcal{D}_j^w$ representation is

$$\mathbb{E}_{\mathcal{D}_j^w}(q,t) = \mathbb{E}_{(b,c)}(q)\, \mathrm{Tr}_{\mathcal{D}_j^w}\left(q^{L_0} z^{J_0^3}\right) \mathrm{Tr}_{\mathcal{N}}\left(q^{L_0}\right). \tag{4.8}$$

In the ghost sector, decomposing $\mathcal{F}_{(b,c)}$ as

$$\mathcal{F}_{(b,c)} = \bigotimes_{n=1}^{\infty} \{|0\rangle_g, b_{-n}|0\rangle_g\} \otimes \{|0\rangle_g, c_{-n}|0\rangle_g\}\,, \tag{4.9}$$

we have

$$\mathbb{E}_{(b,c)}(q) = \mathrm{Tr}_{(b,c)}\left((-1)^{N_g} q^{L_0}\right) = q^{-1} \prod_{n=1}^{\infty}(1-q^n)^2\,, \tag{4.10}$$

and the index can be written as

$$\mathbb{E}_{\mathcal{D}_j^w} = q^{-1} \prod_{n=1}^{\infty}(1-q^n)^2 \times \chi_j^w(q,z) \times \mathrm{Tr}_{\mathcal{N}}\left(q^{L_0}\right). \tag{4.11}$$

Finally, the equality between (4.7) and (4.11) follows immediately from (2.15) and (2.26).

Proceeding similarly for the $\mathcal{C}_{j,\alpha}^w$ representations, we obtain

$$\begin{aligned} \mathbb{S}_{\mathcal{C}_{j,\alpha}^w} &= q^{-1} \prod_{n=1}^{\infty}(1-q^n) \times \sum_{n=-\infty}^{\infty} q^{\frac{(\alpha+n+kw/2)^2}{k}} z^{\alpha+n+kw/2} \hat{\lambda}_{j,\alpha+n}(q) \times \mathrm{Tr}_{\mathcal{N}}\left(q^{L_0}\right), \\ \mathbb{E}_{\mathcal{C}_{j,\alpha}^w} &= q^{-1} \prod_{n=1}^{\infty}(1-q^n)^2 \times \hat{\chi}_{j,\alpha}^w(q,z) \times \mathrm{Tr}_{\mathcal{N}}\left(q^{L_0}\right), \end{aligned} \tag{4.12}$$

and the equality follows again from (2.15) and (2.26).

4.2 Fermionic strings

For NSR strings, the computation of the traces is greatly simplified by decomposing the AdS_3 sector at level k into a purely bosonic $\mathrm{SL}(2,\mathbb{R})$ model at level $k+2$ and three free fermions ψ^a $(a=1,2,3)$ (See section 2.3). From (2.27), we see that the action of $\mathcal{C}$ on the free fermions is

$$\mathcal{C}\psi_n^a\mathcal{C} = (-1)^{\delta_{a,3}}\,\psi_n^a(z)\,. \tag{4.13}$$

The total Hilbert space $\mathcal{F}$ can then be expressed as

$$\mathcal{F} = \mathcal{F}_{(b,c)}\otimes\mathcal{F}_{(\beta,\gamma)}\otimes\mathcal{F}_{\mathrm{SL}(2,\mathbb{R})}\otimes\mathcal{F}_{\psi^a}\otimes\mathcal{F}_{\mathcal{N}}\,. \tag{4.14}$$

Note that the traces on $\mathcal{F}_{(b,c)}\otimes\mathcal{F}_{\mathrm{SL}(2,\mathbb{R})}\otimes\mathcal{F}_{\mathcal{N}}$ can be taken from the above bosonic computation, including there the fermionic sectors of $\mathcal{N}$. We will compute the traces for $\mathcal{D}_j^w$ representations, the $\mathcal{C}_{j,\alpha}^w$ case being identical.

Neveu-Schwarz sector. Using the decomposition (B.5), and (4.13), we find

$$\mathbb{S}_{(\beta,\gamma)}(q) = \mathrm{Tr}_{(\beta,\gamma)}\left(\mathcal{C}q^{L_0}\right) = \prod_{r=\frac{1}{2}}^{\infty}\frac{q^{\frac{1}{2}}}{(1-q^r)(1+q^r)}\,, \tag{4.15}$$

$$\mathbb{S}_{\psi^a}(q) = \mathrm{Tr}_{\psi^a}\left(\mathcal{C}q^{L_0}\right) = \prod_{r=\frac{1}{2}}^{\infty}(1-q^r)(1+q^r)^2\,. \tag{4.16}$$

The total signature is then,

$$\mathbb{S}^{NS}(q,z) = \mathbb{S}_{\mathcal{D}_j^w}(q,z)\mathbb{S}_{(\beta,\gamma)}(q)\mathbb{S}_{\psi^a}(q) = q^{\frac{1}{2}}\times\mathbb{S}_{\mathcal{D}_j^w}(q,z)\times\prod_{r=\frac{1}{2}}^{\infty}(1+q^r)\,. \tag{4.17}$$

To find $\mathbb{E}_{(\beta,\gamma)}$, we decompose

$$\mathcal{F}_{(\beta,\gamma)} = \bigotimes_{r=1/2}^{\infty}\bigoplus_{N_r=0}^{\infty}\left\{\beta_{-r}^{N_r}|0\rangle_g\right\}\otimes\bigotimes_{r=1/2}^{\infty}\bigoplus_{N_r=0}^{\infty}\left\{\gamma_{-r}^{N_r}|0\rangle_g\right\}, \tag{4.18}$$

and then

$$\mathbb{E}_{(\beta,\gamma)}(q) = \mathrm{Tr}_{(\beta,\gamma)}\left((-1)^{N_g}q^{L_0}\right) = \prod_{r=\frac{1}{2}}^{\infty}\frac{q^{\frac{1}{2}}}{(1+q^r)^2}\,. \tag{4.19}$$

We also need

$$\mathrm{Tr}_{\psi^a}\left(q^{L_0}\right) = \prod_{r=\frac{1}{2}}^{\infty}(1+q^r)^3\,. \tag{4.20}$$

So the total index is

$$\mathbb{E}^{NS}(q,z) = \mathbb{E}_{\mathcal{D}_j^w}(q,z)\mathbb{E}_{(\beta,\gamma)}(q)\,\mathrm{Tr}_{\psi^a}\left(q^{L_0}\right) = q^{\frac{1}{2}}\times\mathbb{E}_{\mathcal{D}_j^w}(q,z)\times\prod_{r=\frac{1}{2}}^{\infty}(1+q^r)\,. \tag{4.21}$$

The equality between (4.17) and (4.21) follows then from the bosonic results.

Ramond sector. The traces are computed similarly, giving for the signatures

$$\mathbb{S}_{(\beta,\gamma)}(q) = \prod_{r=1}^{\infty} \frac{q^{\frac{3}{8}}}{(1-q^r)(1+q^r)}\,, \tag{4.22}$$

$$\mathbb{S}_{\psi^a}(q) = q^{\frac{3}{16}} \times \prod_{r=1}^{\infty}(1-q^r)(1+q^r)^2\,, \tag{4.23}$$

and

$$\mathbb{S}^R(q,z) = \mathbb{S}_{\mathcal{D}_j^w}(q,z)\mathbb{S}_{(\beta,\gamma)}(q)\mathbb{S}_{\psi^a}(q) = q^{\frac{9}{16}} \times \mathbb{S}_{\mathcal{D}_j^w}(q,z) \times \prod_{r=1}^{\infty}(1+q^r)\,. \tag{4.24}$$

For the computation of the index, we obtain

$$\mathbb{E}_{(\beta,\gamma)}(q) = \prod_{r=1}^{\infty} \frac{q^{\frac{3}{8}}}{(1+q^r)^2}\,, \tag{4.25}$$

$$\mathrm{Tr}_{\psi^a}\left(q^{L_0}\right) = q^{\frac{3}{16}} \times \prod_{r=1}^{\infty}(1+q^r)^3\,, \tag{4.26}$$

and

$$\mathbb{E}^R(q,z) = \mathbb{E}_{\mathcal{D}_j^w}(q,z)\,\mathbb{E}_{(\beta,\gamma)}(q)\ \mathrm{Tr}_{\psi^a}\left(q^{L_0}\right) = q^{\frac{9}{16}} \times \mathbb{E}_{\mathcal{D}_j^w}(q,z) \times \prod_{r=1}^{\infty}(1+q^r)\,. \tag{4.27}$$

The equality between (4.24) and (4.27) follows again from the bosonic results.

Notice that for the computation of $\mathbb{S}$ in the fermionic case, we have relied on the unitarity of the bosonic $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset, and we did not need the unitarity of the supersymmetric coset, as in the OCQ [20]. This is related to the fact that with the FGZ method we bypass the building of the physical states themselves, which do belong to the *supersymmetric* coset, modulo spurious states.

5. Discussion

The applicability of the FGZ method depends on two points. The first is expressing (a sector of) the matter chiral algebra as a free L_--Virasoro module. This should be checked in each case, although the techniques of [9, 10, 11, 18, 19, 20] can be adapted or adopted for a wide family of backgrounds.

The second point is having enough control over the representations of the worldsheet CFT, so as to compute the traces (4.1). Based on the results for $AdS_3 \times \mathcal{N}$, several related exact CFT backgrounds could be dealt with successfully. In particular, the BTZ black hole [37] (an orbifold of AdS_3), the non-compact spaces studied in [38] and pp-waves backgrounds, obtained as Penrose limits of $AdS_3 \times S^3 \times M^4$ (see [39] and references therein).

An additional class of related backgrounds are gauged WZW models including an $\mathrm{SL}(2,\mathbb{R})$ factor, such as the lorentzian 2D black hole [40, 25] ($\mathcal{M} = \mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$), the cosmological Nappi-Witten model [41] ($\mathcal{M} = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2)/\,\mathrm{U}(1)^2$) and the generalization of the latter studied in [42]. These backgrounds are particularly interesting, because although the matter $\mathcal{M}$ CFT (the relative cohomology of the BRST complex in the gauged WZW model) contains negative-normed states, it cannot be decomposed as $\mathrm{U}(1) \times \mathcal{M}/\,\mathrm{U}(1)$. Here we can use to advantage the cohomological analysis of gauged WZW models in [35, 43], which show that a vanishing theorem holds in G/H (for certain subgroups H). This allows to compute $\mathbb{S}$ and $\mathbb{E}$ on G/H by taking the traces over $G \otimes \mathcal{F}_{\text{ghosts}} \otimes H$, and following a chain of identities similar to those in (3.21) and (3.25). In this way, as we did in this work, the flowed sectors of the $\mathrm{SL}(2,\mathbb{R})$ factor in G can be included. This will be worked out in detail in a future work.

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Note added. After this paper appeared in hep-th, the work [48] appeared, where the methods of [9] are applied to $AdS_3 \times \mathcal{N}$ backgrounds.

A. BRST essentials

We review here the basics of the (b, c) and (β, γ) algebra that we need in this work. More details can be found in [44]. The zero-mode quantization has some subtleties, which have been discussed, *inter alia*, in [45].

A.1 Bosonic strings

The anticommuting (b, c) reparametrization ghosts, with conformal dimensions $(2, -1)$, have the OPEs

$$\begin{aligned} b(z)c(w) &\sim c(z)b(w) \sim \frac{1}{z-w}\,, \\ b(z)b(w) &\sim c(z)c(w) \sim 0\,. \end{aligned} \tag{A.1}$$

The ghost number operator is defined as

$$\begin{aligned} N_g &= -\oint dz\, b(z)c(z)\,, \\ &= \sum_{n=1}^{\infty} (c_{-n}b_n - b_{-n}c_n) + c_0 b_0 - \frac{1}{2}\,. \end{aligned} \tag{A.2}$$

The ordering constant $-1/2$ can be fixed in several ways [44], the most immediate one being the requirement of N_g being antihermitean, i.e., $N_g^\dagger = -N_g$. For this we use $b_n^\dagger = b_{-n}$, $c_n^\dagger = c_{-n}$.

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The charge of the $(b-c)$ modes under N_g is

$$
\begin{aligned}
[N_g, b_m] &= -b_m\,, \\
[N_g, c_m] &= +c_m\,.
\end{aligned}
\tag{A.3}
$$

The zero modes b_0, c_0 satisfy the Clifford algebra $\{b_0, c_0\} = 1$, so there is a double-degenerate vacuum $|\uparrow\rangle$, $|\downarrow\rangle$, which satisfies

$$
\begin{aligned}
&c_0|\downarrow\rangle = |\uparrow\rangle \qquad b_0|\uparrow\rangle = |\downarrow\rangle\,, \\
&c_0|\uparrow\rangle = b_0|\downarrow\rangle = c_n|\uparrow,\downarrow\rangle = b_n|\uparrow,\downarrow\rangle = 0\,, \qquad n > 0\,.
\end{aligned}
\tag{A.4}
$$

On the vacua $|\uparrow\rangle$, $|\downarrow\rangle$ we have $L_0^{(b,c)} = -1$. The zero-modes algebra implies that the inner product on the ghost space is consistently defined by

$$
\begin{aligned}
\langle\uparrow|c_0|\uparrow\rangle &= \langle\uparrow|c_0|\uparrow\rangle = 1\,, \\
\langle\uparrow|c_0|\downarrow\rangle &= 0\,,
\end{aligned}
\tag{A.5}
$$

and from this follows the inner product of any excited state.

The BRST charge

$$
Q = \oint dz c(z)\left(T^M(z) + \frac{1}{2}T^{(b,c)}(z)\right)\,,
\tag{A.6}
$$

is an hermitean operator which carries ghost number $+1$, and $Q^2 = 0$ if $c_M = 26$. Here T^M is the matter stress tensor. In this work we consider the restriction to the space $\mathcal{F}$ defined by $b_0 = L_0 = 0$, where L_0 is the zero mode of $T = T^M + T^{(b,c)}$. This chooses the $|\downarrow\rangle$ vacuum, which will be denoted $|0\rangle_g$. In $\mathcal{F}$ we redefine N_g to be in $\mathbb{Z}$ by assigning ghost number 0 to $|0\rangle_g$ and ignoring the last two terms in the expansion (A.2). The action of Q is consistent with the restriction to $\mathcal{F}$ because $\{Q, b_0\} = L_0$.

A.2 Fermionic strings

For NSR strings, we have, on top of the (b,c) system, the super-reparametrization ghosts (β,γ), with conformal dimensions $(3/2, -1/2)$ and OPEs

$$
\begin{aligned}
\gamma(z)\beta(w) &\sim \frac{1}{z-w}\,, \\
\beta(z)\gamma(w) &\sim \frac{-1}{z-w}\,, \\
\beta(z)\beta(w) &\sim \gamma(z)\gamma(w) \sim 0\,.
\end{aligned}
\tag{A.7}
$$

The modes of (β,γ) satisfy $\beta_r^\dagger = -\beta_{-r}$, $\gamma_r^\dagger = \gamma_{-r}$, where $r \in \mathbb{Z}$ ($r \in \mathbb{Z} + 1/2$) for the Ramond (Neveu-Schwarz) sector. In the Ramond sector, the zero modes again make the vacuum double-degenerate. We choose $\beta_0 = 0$ in $\mathcal{F}$, and denote the selected vacuum also by $|0\rangle_g$. We have then,

$$
\beta_{r\geq 0}|0\rangle_g = \gamma_{r>0}|0\rangle_g = 0\,.
\tag{A.8}
$$

On $|0\rangle_g$, we have $L_0^{(\beta,\gamma)} = 3/8$ ($L_0^{(\beta,\gamma)} = 1/2$) for the Ramond (Neveu-Schwarz) sector. The $|0\rangle_g$ ghost number is zero, and the operator N_g in (A.2) is extended to satisfy

$$\begin{aligned} [N_g, \beta_r] &= -\beta_r \,, \\ [N_g, \gamma_r] &= +\gamma_r \,. \end{aligned} \tag{A.9}$$

The BRST charge is now

$$Q = \oint dz \left[c(z) \left(T^M(z) + \frac{1}{2} T^g(z) \right) + \gamma(z) \left(G^M(z) + \frac{1}{2} G^g(z) \right) \right] , \tag{A.10}$$

with $Q^2 = 0$ if $c_M = 10$. Here G^M is the matter supercurrent, $T^g = T^{(b,c)} + T^{(\beta,\gamma)}$ and G^g the supercurrent of the superconformal combined $(b,c) - (\beta,\gamma)$ systems. The $L_0 = 0$ condition is imposed with $T^M + T^g$. The space $\mathcal{F}$ is again defined by $b_0 = L_0 = 0$. In the Ramond sector, also by $\beta_0 = G_0 = 0$, where G_0 is the zero mode of $G = G^M + G^g$. This restriction is consistent because $\{Q, \beta_0\} = G_0$.

B. The action of $\mathcal{C}$ in the ghost sector

In this appendix we will show that the operator $\mathcal{C}$, defined in section 2, interchanges between b_n and c_n (and β_r and γ_r) modes, thus mapping $\mathcal{F}_g$ to $\mathcal{F}_{-g}$.

We can decompose the ghost sector $\mathcal{F}_{(b,c)}$ into

$$\mathcal{F}_{(b,c)} = \bigotimes_{n=1}^{\infty} \{|0\rangle_g, |+,n\rangle\} \otimes \{|0\rangle_g, |-,n\rangle\} \,, \tag{B.1}$$

where

$$|\pm, n\rangle \equiv \frac{(b_{-n} \pm c_{-n})}{\sqrt{2}} |0\rangle_g \,, \tag{B.2}$$

satisfy

$$\begin{aligned} \langle \pm, n | \pm, m \rangle &= \pm \delta_{n,m} \,, \\ \langle +, n | -, m \rangle &= 0 \,, \end{aligned} \tag{B.3}$$

so from $\mathcal{C}|\pm, m\rangle = \pm|\pm, m\rangle$ we have

$$\begin{aligned} \mathcal{C} c_n \mathcal{C} &= b_n \,, \\ \mathcal{C} b_n \mathcal{C} &= c_n \,. \end{aligned} \tag{B.4}$$

For fermionic strings, we use the decomposition

$$\mathcal{F}_{(\beta,\gamma)} = \bigotimes_{r>0} \bigoplus_{N_r=0}^{\infty} \{|+, N_r\rangle\} \otimes \bigotimes_{r>0} \bigoplus_{N_r=0}^{\infty} \{|-, N_r\rangle\} \,, \tag{B.5}$$

where

$$|\pm, N_r\rangle \equiv \frac{(\beta_{-r} \pm \gamma_{-r})^{N_r}}{\sqrt{2 N_r!}} |0\rangle_g \,, \tag{B.6}$$

satisfy

$$\begin{aligned} \langle \pm, N_r | \pm, N_s \rangle &= (\pm 1)^{N_r} \delta_{r,s} \delta_{N_r,N_s} \,, \\ \langle +, N_r | -, N_s \rangle &= 0 \,, \end{aligned} \tag{B.7}$$

so we have

$$\begin{aligned} \mathcal{C} \beta_r \mathcal{C} &= (-1)^{S(r)} \gamma_r \,, \\ \mathcal{C} \gamma_r \mathcal{C} &= (-1)^{S(r)} \beta_r \,, \end{aligned} \tag{B.8}$$

where $S(r)$ equals $0\,(1)$ if $r < 0\,(r > 0)$.

C. $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1) \times \mathrm{U}(1)$ character decomposition

In this appendix we will prove explicitly the character decompositions (2.14) and (2.26). Let us start with the representation $\mathcal{D}_j^+$, whose character is

$$\chi_j^+(q,z) = \frac{q^{-\frac{j(j-1)}{k-2}} z^j}{\prod_{n=1}^\infty (1-q^n)(1-q^{n-1}z)(1-q^n z^{-1})} \,. \tag{C.1}$$

We will use the technique of infinite partial fractions, which has been used in [46, 47] in order to prove similar identities. Define

$$f_N(q,z) = \frac{1}{\prod_{n=1}^N (1-q^{n-1}z)\,(1-q^n z^{-1})} \,, \tag{C.2}$$

which can be decomposed into partial fractions as

$$f_N(q,z) = \sum_{k=1}^N \frac{a_k(q)}{(1-q^{k-1}z)(1-q^k z^{-1})} \,. \tag{C.3}$$

The coefficients $a_k(q)$ are

$$\begin{aligned} a_k(q) &= \lim_{z\to q^k} (1-q^{k-1}z)(1-q^k z^{-1}) f_N(q,z) \,, \\ &= \frac{1-q^{2k-1}}{\prod_{n=k}^{N+k-1}(1-q^n) \prod_{n=1 n\neq k}^N (1-q^{n-k})} \,, \\ &= \frac{(1-q^{2k-1})(-1)^{k-1} q^{\frac{k(k-1)}{2}}}{\prod_{n=1}^{N+k-1}(1-q^n) \prod_{n=1}^{N-k}(1-q^n)} \,, \end{aligned} \tag{C.4}$$

thus for $N \to \infty$ we have

$$\begin{aligned} \lim_{N\to\infty} f_N(q,z) &= \frac{1}{\prod_{n=1}^\infty (1-q^{n-1}z)(1-q^n z^{-1})} \,, \\ &= \frac{1}{\prod_{n=1}^\infty (1-q^n)^2} \left[\sum_{k=1}^\infty \frac{(1-q^{2k-1})(-1)^{k-1} q^{\frac{k(k-1)}{2}}}{(1-q^{k-1}z)(1-q^k z^{-1})} \right] . \end{aligned} \tag{C.5}$$

The denominators in the sum of (C.5) can be expanded and resummed as

$$
\begin{aligned}
\frac{1}{(1-q^{k-1}z)(1-q^{k}z^{-1})} &= \sum_{m,n=0}^{\infty} q^{m(k-1)} z^{m} q^{nk} z^{-n}\,, \\
&= \frac{1}{(1-q^{2k-1})}\left[\sum_{n=0}^{\infty} q^{n(k-1)} z^{n} + \sum_{n=-\infty}^{-1} q^{-nk} z^{n}\right]. \qquad \text{(C.6)}
\end{aligned}
$$

Thus (C.5) can be written as

$$
\begin{aligned}
\frac{1}{\prod_{n=1}^{\infty}(1-q^{n-1}z)(1-q^{n}z^{-1})} &= \frac{1}{\prod_{n=1}^{\infty}(1-q^{n})^{2}} \times \\
&\quad \times \Bigg[\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} z^{n}(-1)^{k-1} q^{\frac{1}{2}(k+2n)(k-1)} + \\
&\quad + \sum_{n=-\infty}^{-1}\sum_{k=1-2n}^{\infty} z^{n}(-1)^{k-1} q^{\frac{1}{2}(k+2n)(k-1)}\Bigg], \qquad \text{(C.7)}
\end{aligned}
$$

where in the second term we have changed the index k to $k+2n$ in order to make the exponents of q be the same in both terms. Now, using that for $n \in \mathbb{Z}^{-}$

$$
\sum_{k=1}^{-2n}(-1)^{k-1} q^{\frac{1}{2}(k+2n)(k-1)} = 0\,, \qquad \text{(C.8)}
$$

and changing the index k to $s = k-1$ we obtain finally

$$
\frac{1}{\prod_{n=1}^{\infty}(1-q^{n-1}z)(1-q^{n}z^{-1})} = \frac{1}{\prod_{n=1}^{\infty}(1-q^{n})^{2}} \sum_{n=-\infty}^{\infty}\sum_{s=0}^{\infty} z^{n}(-1)^{s} q^{\frac{1}{2}s(s+2n+1)}\,, \qquad \text{(C.9)}
$$

and multiplying both sides by $q^{-\frac{j(j-1)}{k-2}} z^{j} \prod_{n=1}^{\infty}(1-q^{n})^{-1}$, the first line of (2.14) follows.

From (2.22), we see that the flowed characters can be obtained from $\chi_j^{+}(q,z)$ by replacing $z \to zq^{-w}$, and multiplying the resulting expression by $z^{kw/2} q^{-kw^2/4}$. So we have,

$$
\begin{aligned}
\chi_j^{w}(q,z) &= \frac{q^{-\frac{j(j-1)}{k-2} - kw^2/4} z^{kw/2}}{\prod_{n=1}^{\infty}(1-q^{n})^{3}} \sum_{n=-\infty}^{\infty}\sum_{s=0}^{\infty} (zq^{-w})^{n+j} (-1)^{s} q^{\frac{1}{2}s(s+2n+1)}\,, \\
&= \frac{q^{-\frac{j(j-1)}{k-2}} z^{j+kw/2}}{\prod_{n=1}^{\infty}(1-q^{n})^{3}} \sum_{n=-\infty}^{\infty}\sum_{s=0}^{\infty} q^{\frac{(j+n)^2}{k} - \frac{(j+n+kw/2)^2}{k}} z^{n} (-1)^{s} q^{\frac{1}{2}s(s+2n+1)}\,, \qquad \text{(C.10)}
\end{aligned}
$$

and this identity is the first line of (2.26).

The decompositions for $\mathcal{C}_{j,\alpha}$ representations are immediate and are left as an exercise.

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